Ergod. Th. & Dynam. Sys. (2004), **24**, 1277–1281
DOI: 10.1017/S0143385704000409

L^2 regularity of measurable solutions of a finite-difference equation of the circle†

MICHAEL ROBERT HERMAN

Mathematics Institute, Warwick University, UK
and
Centre de Mathematiques, Ecole Polytechnique, Plateau de Palaiseau, 91120 Palaiseau, France

We show that if φ is a lacunary Fourier series and the equation $\psi(x) - \psi(x+\alpha) = \varphi(x)$, x mod 1 has a measurable solution φ, then in fact the equation has a solution in L^2.

(1) We consider the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and the translations (or rotations) $R_\alpha = x \to x + \alpha (\alpha \in \mathbb{T})$.

For $1 \le p \le +\infty$, let $L^p = L^p(\mathbb{T}, dx, \mathbb{C})$ with the norm $\|\cdot\|_p$. The only measure considered is the Haar measure of $\mathbb{T}$, $dx = m$. All equalities are to be considered m-almost everywhere.

(2) Let $\varphi \in L^1$ and $\alpha \in \mathbb{T}$; we try to solve

$$\psi - \psi \circ R_\alpha = \varphi \tag{*}$$

with ψ measurable and the equality almost everywhere.

If one supposes that ψ is in L^1, then by identification of Fourier coefficients if

$$\varphi(x) = \sum_{k \in \mathbb{Z}} \hat{\varphi}(k) e^{2xikx},$$

then one has

$$\psi(x) = \sum_{k \in \mathbb{Z}} \frac{\hat{\varphi}(k)}{1 - e^{2\pi i k \alpha}} e^{2\pi i k x},$$

(with the convention that $0/0 = 0$). (Of course one has $0 = \int_{\mathbb{T}} \varphi(x)\, dx$).

(3) The case when $a = p/q$ (mod 1), $(p, q) = 1$. Then a necessary and sufficient condition for measurable solutions to (*) is

$$\sum_{i=0}^{q-1} \varphi \circ R_{i\alpha} = 0. \tag{1}$$

If (1) is satisfied then the equation (*) has solutions just as regular as is φ.

† This work of Michel Herman appeared only as a preprint of the Mathematics Institute, University of Warwick, dated May 1976. It was turned into TEX format by Claire Desescures. Minor editorial work was done by Albert Fathi.

(4) The case when α is irrational. It is easy (by Fourier series) to construct $\varphi \in L^1$ with $\int_{\mathbb{T}} \varphi(x)\, dx = 0$ and an irrational α such that the equation (*) has no solution in L^1. By the ergodicity of R_α, measurable solutions of (*) differ by a constant.

If one looks for solutions of (*) which are only measurable then Anosov has shown that one has necessarily

$$\int_{\mathbb{T}} \varphi(x)\, dx = 0 \quad (\text{for } \varphi \in L^1).$$

Furthermore, Anosov has constructed $\varphi \in C^\omega(\mathbb{T})$ with $\int_{\mathbb{T}} \varphi(x)\, dx = 0$ and an irrational α such that

$$\sup_{k \neq 0} \left| \frac{\hat{\varphi}(k)}{1 - e^{2\pi i k \alpha}} \right| = +\infty,$$

but nevertheless the equation (*) has a measurable solution ψ (of course not in L^1) (see [**1**]).

We will show that the examples of Anosov cannot happen when φ is a lacunary Fourier series.

It is then easy to construct a φ with $\int_{\mathbb{T}} \varphi(x)\, dx = 0$ and an irrational α such that the equation (*) has no measurable solution ψ (since there is no L^2 solution).

For other examples see [**6**].

(5) Let $\Lambda_+ = n_i$ be a lacunary sequence of positive integers: $n_0 = 1$ and $n_{n+1}/n_i \geq q > 1$ for all i.

Let $\Lambda = \Lambda_+ \cup \{0\} \cup (-\Lambda_+)$ be the symmetric sequence of integers.

One denotes

$$L^p_\Lambda = \{\varphi \in L^p \mid \hat{\varphi}(n) = 0 \text{ if } n \notin \Lambda\}.$$

One says that $\varphi \in L^1$ is a lacunary Fourier series if there exists a lacunary sequence Λ as above such that $\varphi \in L^1_\Lambda$. Then one has, for all $1 \leq p < +\infty$, $\varphi \in L^p_\Lambda$; and all the norms $\|\cdot\|_p$ are equivalent on L^2_Λ (see [**5**]).

(6) We propose to prove the following.

THEOREM. *Let $\varphi \in L^2_\Lambda$ and $\alpha \in \mathbb{T}$. If the equation*

$$(*) \psi - \psi \circ R_\alpha = \varphi$$

has a measurable solution ψ, then the equation has a solution in L^2_Λ and if $\alpha \in \mathbb{T} - \mathbb{Q}/\mathbb{Z}$ then in fact, by the ergodicity of R_α, $\psi \in L^2_\Lambda$.

To prove the theorem one needs the following lemmas.

(7)

LEMMA. *Let $f : \mathbb{T} \to \mathbb{T}$ be a bijection preserving the Haar measure m.*

Let K be a measurable set of $\mathbb{T}$. Let $\epsilon > 0$ and the set of integers

$$A = \{n \in \mathbb{Z} \mid m(K \cap f^n(K)) \geq m(K)^2 - \epsilon\}.$$

The set of integers A is relatively dense: there exists a positive integer k, such that $\{j, \ldots, j+k\} \cap A \neq \phi$, for all $j \in \mathbb{Z}$.

For a proof see [**3**, p. 31].

(8)

LEMMA†. Let L^2_Λ be given. There exist constants $C > 0$ and b $(0 < b < 1)$ such that if $B \subset \mathbb{T}$ is measurable with $m(B) \geq b$, then for all $\varphi \in L^2_\Lambda$ one has

$$C\left(\int_B |\varphi(x)|^2\,dx\right)^{1/2} \geq \|\varphi\|_2.$$

Proof. Let $0 < a < 1$ and $\varphi \in L^2_\Lambda$ with $\|\varphi\|_2 = 1$. Let

$$A(\varphi) \equiv A = \{x \in \mathbb{T} \mid |\varphi(x)| \geq a\}.$$

We have $\|\varphi\|_2^2 = 1 = \int_{\mathbb{T}-A} |\varphi(x)|^2\,dx + \int_A |\varphi(x)|^2\,dx \leq a^2 + \int_A |\varphi(x)|^2\,dx$.

One has by the Hölder inequality

$$1 \leq \|\varphi\|_4 (m(A))^{1/4} + a.$$

Since the norms $\|\cdot\|_2$ and $\|\cdot\|_4$ are equivalent on L^2_Λ, one has $\|\cdot\|_4 \leq k\|\cdot\|_2$, k being a constant greater than 1.

It follows that

$$m(A) \geq \left(\frac{1-a}{k}\right)^4; \tag{2}$$

choose

$$b = 1 - \frac{1}{2}\left(\frac{1-a}{k}\right)^4.$$

If $B \subset \mathbb{T}$ with $m(B) \geq b$ and if $\varphi \in L^2_\Lambda$ with $\|\varphi\|_2 = 1$, we have

$$m(A(\varphi) \cap B) \geq \frac{1}{2}\left(\frac{1-a}{k}\right)^4$$

by (2), so

$$\int_B |\varphi(x)|^2\,dx \geq \frac{1}{2}a^2\left(\frac{1-a}{k}\right)^4 = \left(\frac{1}{C}\right)^2.$$

The result follows by

$$C\left(\int_B |\varphi(x)|^2\,dx\right)^{1/2} \geq \|\varphi\|_2. \qquad \square$$

(9)

LEMMA. *Let $\varphi \in L^2$. A necessary and sufficient condition for a $\psi \in L^2$ that verifies $\psi - \psi \circ R_\alpha = \varphi$ to exist is that $\sup_{n\in N} \|\varphi_n\|_2 < +\infty$ with $\varphi_n = \sum_{i=0}^{n-1} \varphi \circ R_{i\alpha}$.*

For the proof see [**4**]. In fact it results from the more general lemma, which uses the fact that the unit ball of a reflexive Banach space is weakly compact, and the Markov–Kakutani fixed point theorem (affine version).

† I thank Y. Meyer who brought to my attention the fact that Carleson has proved a stronger lemma (unfortunately unpublished): For every B with $m(B) > 0$ there exists $C(m(B), q) > 0$ such that one has the conclusion of the lemma. I thank B. Maurey for the proof proposed.

LEMMA. *Let L be a reflexive Banach space of norm $\|\cdot\|$ and $u : L \to L$ a continuous linear operator. Given $x \in L$, a sufficient condition for the existence of a $y \in L$ satisfying $y - u(y) = x$ to exist is that*

$$\sup_{n\in N}\left\|\sum_{i=0}^{n-1} u^i(x)\right\| < +\infty;$$

the condition is necessary if $\sup_{n\in N}\|u^n\| < +\infty$.

(10) *Proof of the theorem.* Let L^2_Λ be given and be determined by item (8) (and that depends on Λ).

Let $\epsilon > 0$ with $(1-\epsilon)^2 - \epsilon \geq b$.

One starts with a measurable solution of

$$\psi - \psi \circ R_\alpha = \varphi, \tag{*}$$

with $\varphi \in L^2_\Lambda$. There exists a compact set $K \subset \mathbb{T}$ of measure $\geq 1-\epsilon$, such that $\psi_{|K}$ is continuous. By (*) one has

$$\psi - \psi \circ R_{n\alpha} = \sum_{i=0}^{n-1} \varphi \circ R_{i\alpha} \equiv \varphi_n.$$

It follows that

$$\left(\int_{K\cap R_{n\alpha}(K)} |\varphi_n(x)|^2\, dx\right)^{1/2} \leq 2\sup_{x\in K}|\psi(x)| < +\infty.$$

Let $A = \{n \in \mathbb{Z} \mid m(K \cap R_{n\alpha}(K)) \geq (1-\epsilon)^2 - \epsilon \geq b\}$. By item (7), the subset A is a relatively dense sequence of integers, and let k be the integer of (7). Let $B = \{-k, -k+1, \dots, k\}$. Since $\varphi_n \in L^2_\Lambda$ by (8) one has

$$\sup_{n\in A}\|\varphi_n\|_2 = C_1 < +\infty.$$

Let $C_2 = \sup_{n\in B}\|\varphi_n\|_2 < +\infty$. Since every $n \in \mathbb{Z}$ can be written as $n = n_1 + n_2$ with $n_1 \in A$ and $n_2 \in B$ and if n_1, and n_2 are positive integers, we have

$$\varphi_{n_1+n_2} = \varphi_{n_1} \circ R_{n_2\alpha} + \varphi_{n_2};$$

finally we deduce that

$$\sup_{n\in\mathbb{Z}}\|\varphi_n\|_2 \leq C_1 + C_2$$

and the theorem results from (9).

(11) From the theorem we deduce the following: if $\varphi \in L^2_\alpha$, α is irrational, and ψ is measurable and satisfies $\psi - \psi \circ R_\alpha = \varphi$, then $\psi \in L^p$ for every $1 \leq p < +\infty$ since ψ is a lacunary Fourier series. In general, $\psi \notin L^\infty$ even if φ is of class C^ω as we will show by a classical example.

Construction of an irrational α. Let $\alpha = 1/(a_1 + (1/(a_2 + \cdots)))$ be the continued fraction of an irrational α ($a_i \geq 1, a_i \in \mathbb{N}$).

If p_n/q_n are the convergents of α, one has $q_0 = 1, q_1 = a_1$ and $q_n = a_n q_{n-1} + q_{n-2}$, if $n \geq 2$. If $x \in \mathbb{R}$ and $|||x|||$ is the distance of x to the nearest integer, one has

$$|||q_n\alpha||| < \frac{1}{q_{n+1}} \leq \frac{1}{a_{n+1}q_n}.$$

If one chooses the sequence (a_i) so that it increases sufficiently rapidly, one easily constructs an irrational α such that, for every $n \geq 2$, one has

$$|||q_n\alpha||| \leq e^{-q_n}. \tag{+}$$

Let us remark that, for every irrational α, $(q_{2n})_{n\in\mathbb{N}}$ is a lacunary sequence of positive integer (in fact we have $q_{2n+2}/q_{2n} \geq 2$ and also $q_{2n+1}/q_{2n-1} \geq 2$).

Construction of φ. Let $n \geq 1$ be a sequence of complex numbers satisfying

$$\sum_{n=1}^{\infty} |c_{2n}|^2 < +\infty \quad \text{but} \quad \sum_{n=1}^{\infty} |c_{2n}| = +\infty.$$

Let $\varphi(x) = \sum_{n=1}^{\infty} c_{2n}(1 - e^{2\pi i q_{2n}\alpha})e^{2\pi i q_{2n}x}$.

If α satisfies (+), then $\varphi \in C^{\omega}(\mathbb{T}, \mathbb{C})$ (and one has $0 = \int_{\mathbb{T}} \varphi(x)\,dx$).

Let $\psi(x) = \sum_{n=1}^{\infty} c_{2n}e^{2\pi i q_{2n}x}$; one has $\psi \in L^2$ (and ψ is a lacunary Fourier series). Furthermore, one has

$$\psi - \psi \circ R_\alpha = \varphi.$$

But $\psi \notin L^\infty$, for if this was the case then, since ψ is a lacunary Fourier series, we would have $\sum_{n=1}^{\infty} |c_{2n}| < +\infty$, which is contrary to the choice of the sequence (c_{2n}) (see [**5**]).

(12) We have shown a proposition in [**2**] that implies the following remark.

Remark. Let $\varphi : \mathbb{T} \to \mathbb{R}$ be continuous (but not necessarily lacunary) and α irrational. We suppose that there exists $\psi \in L^\infty$ with $\psi - \psi \circ R_\alpha = \varphi$; then ψ is almost everywhere equal to a continuous function.

References

[**1**] D. V. Anosov. On an additive functional homology equation connected with an ergodic rotation of the circle. *Translation Math. U.S.S.R. Izvestija* **74** (1973), 1257–1271.

[**2**] M. R. Herman. Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations. *Thesis*, Orsay 1976.

[**3**] K. E. Petersen. Introductory ergodic theory. *Lecture Notes*, University of North Carolina at Chapel Hill.

[**4**] K. E. Petersen. On a series of cosecants related to a problem in ergodic theory. *Compositio Math.* **26** (1973), 313–317.

[**5**] A. Zygmund. *Trigonometric Series*. Cambridge University Press, 1959.

[**6**] A. Y. Gordon. *Funktsional'ny Analiz Evo Prilozheniya* **9**(4) (1975), 71–72 (in Russian).